SAT-Based Subsumption Resolution

CADE29

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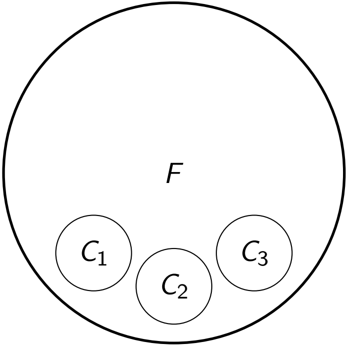
Saturation in FOL Theorem Proving



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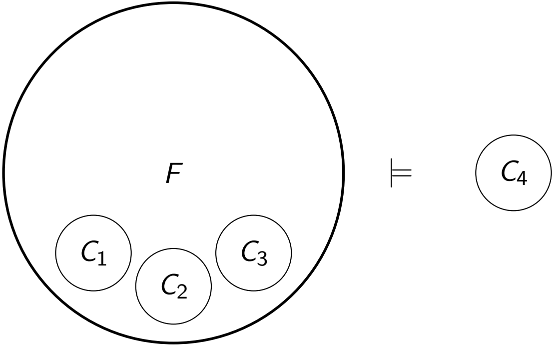

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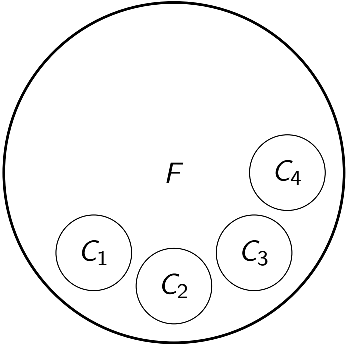

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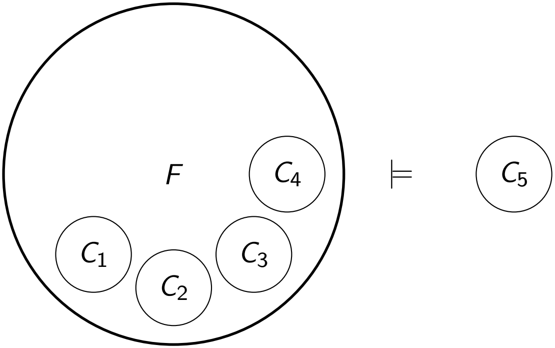

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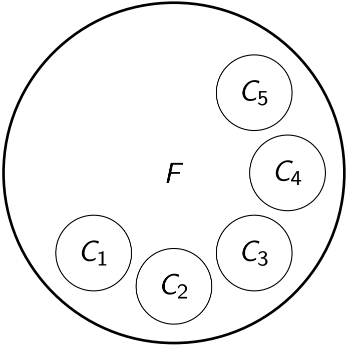

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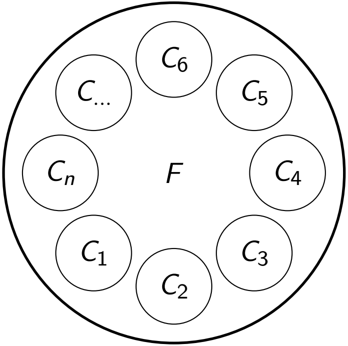

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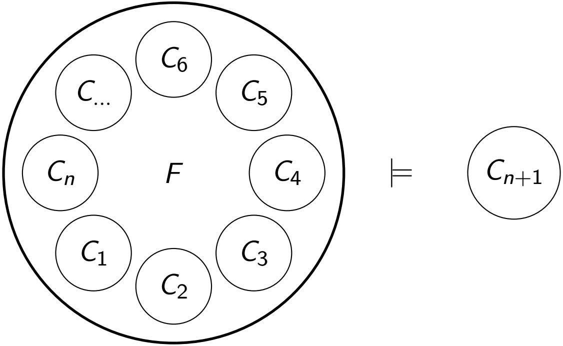

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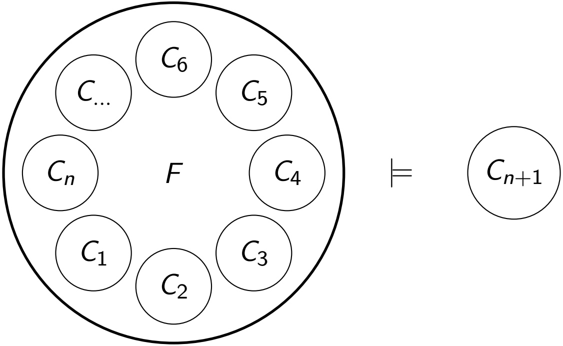

Out of memory!

Subsumption

Definition

A clause L *subsumes* a distinct clause M iff there is a substitution σ such that

$$\sigma(L) \subseteq^* M$$

where $\subseteq^*$ is the sub-multiset inclusion relation.

If L subsumes M, then M is redundant and can be removed from the formula.

Subsumption - Examples

Example (propositional logic)

$$L = a \vee b$$
$$M = a \vee b \vee c$$

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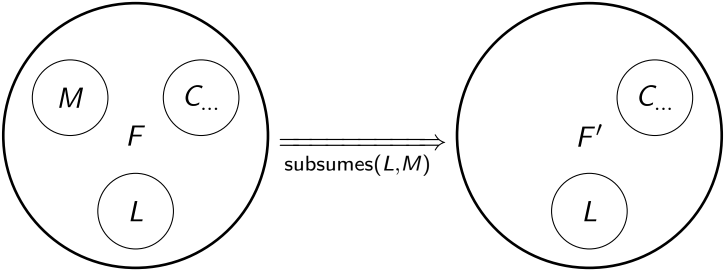
Example (FOL)

$$L = p(x_1, x_2) \vee p(f(x_2), x_3)$$
$$M = \neg p(f(c), d) \vee p(f(y), c) \vee p(f(c), g(d))$$

L subsumes M with the substitution $\sigma = \{x_1 \mapsto f(y), x_2 \mapsto c, x_3 \mapsto g(d)\}$.

Subsumption - Intuition



Subsumption Resolution

Resolution (Simplified)

$$\frac{L^* \vee l' \qquad \neg\sigma(l') \vee M^*}{\sigma(L^*) \vee M^*}$$

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$$\frac{L^* \vee l' \qquad \neg\sigma(l') \vee M^*}{\sigma(L^*) \vee M^*}$$

Definition

Clauses M and L are said to be the main and side premise of subsumption resolution, respectively, iff there is a substitution σ, a set of literals $L' \subseteq L$ and a literal $m' \in M$ such that

$$\sigma(L') = \{\neg m'\} \quad \text{and} \quad \sigma(L \setminus L') \subseteq M \setminus \{m'\}.$$

Subsumption Resolution aims to remove a literal from the main premise.

Subsumption Resolution - Example 1

Example (propositional logic)

$$\frac{L := \boxed{a} \vee b \qquad M := \boxed{\neg a} \vee b \vee c}{M^* := b \vee c}$$

$\neg a$ is the resolution literal. M^* subsumes M and can replace M in the clause set.

Subsumption Resolution - Example 1

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$$\frac{L := \boxed{a} \vee b \qquad \cancel{M := \boxed{\neg a} \vee b \vee c}}{M^* := b \vee c}$$

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Subsumption Resolution - Example 2

Example (FOL)

$$L = p(x_1, x_2) \vee p(f(x_2), x_3)$$
$$M = \neg p(f(y), d) \vee p(g(y), c) \vee \neg p(f(c), e)$$
$$\sigma = \{x_1 \mapsto g(y), x_2 \mapsto c, x_3 \mapsto e\}$$

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$$\dfrac{\dfrac{p(x_1, x_2) \vee p(f(x_2), x_3)}{p(g(y), c) \vee \boxed{p(f(c), e)}} \qquad \neg p(f(y), d) \vee p(g(y), c) \vee \boxed{\neg p(f(c), e)}}{M^* := \neg p(f(y), d) \vee p(g(y), c)}$$

Subsumption Resolution - Example 2

Example (FOL)

$$L = p(x_1, x_2) \lor p(f(x_2), x_3)$$
$$M = \neg p(f(y), d) \lor p(g(y), c) \lor \neg p(f(c), e)$$
$$\sigma = \{x_1 \mapsto g(y), x_2 \mapsto c, x_3 \mapsto e\}$$

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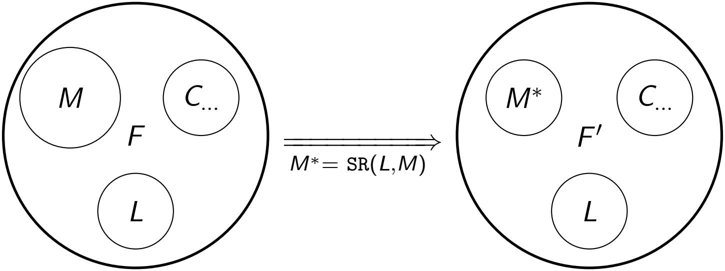
Subsumption Resolution - Intuition



Importance of Redundancy Elimination

```
$ vampire Problems/GRP/GRP140-1.p -fsr off -t 30
...
132544. $ false
% Termination reason: Refutation
% Memory used [KB]: 308054
% Time elapsed: 6.654 s
------------------------------------------------------------
```

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% Termination reason: Refutation
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% Time elapsed: 6.654 s
------------------------------------------------------------
$ vampire Problems/GRP/GRP140-1.p -fsr on -t 30
...
4918. $ false
% Termination reason: Refutation
% Memory used [KB]: 12025
% Time elapsed: 0.150 s
```

Relevance of Speed

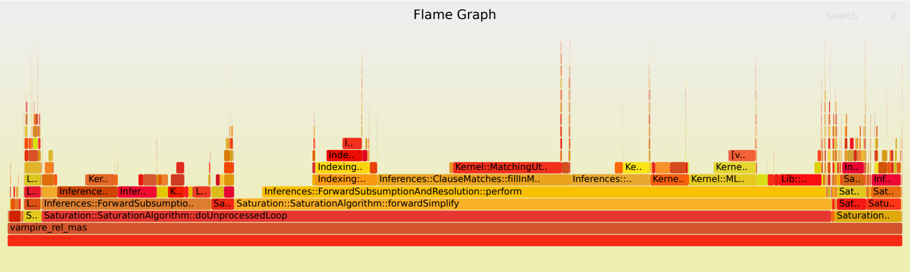

Figure: Typical profiling results for a TPTP problem (GRP001+6).

Building upon Previous Work

Previous Work

[Rath et al., 2022] introduced a SAT-based subsumption procedure.

- Encode subsumption as SAT problem.
- Tailor SAT solver to reason over substitutions.
- Use SAT solver to find a suitable substitution for subsumption.

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Our Contribution

We build upon the work of [Rath et al., 2022].

- Introduce constraints for subsumption resolution.
- Convert subsumption resolution to SAT problem.
- Integrate subsumption and subsumption resolution in Vampire.
- Optimize the simplifying loop of Vampire.

From Definition to Constraints

Theorem (Subsumption Resolution Constraints)

Clauses M and L are the main and side premise, respectively, of an instance of the subsumption resolution rule $\mathcal{SR}$ iff there exists a substitution σ that satisfies the following four properties:

existence $\exists i\, j.\, \sigma(l_i) = \neg m_j$

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completeness	$\forall i.\, \exists j.\, \big(\sigma(l_i) = \neg m_j \vee \sigma(l_i) = m_j\big)$

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completeness	$\forall i.\, \exists j.\, \big(\sigma(l_i) = \neg m_j \vee \sigma(l_i) = m_j\big)$
coherence	$\forall j.\, \big(\exists i.\, \sigma(l_i) = m_j \Rightarrow \forall i.\, \sigma(l_i) \neq \neg m_j\big)$

SAT variables

Let

$$L = \{l_1, \dots, l_{|L|}\} \quad M = \{m_1, \dots, m_{|M|}\}$$

We define the following SAT variables:

- $b^+_{i,j} \Leftrightarrow \sigma(l_i) = m_j$
- $b^-_{i,j} \Leftrightarrow \sigma(l_i) = \neg m_j$

This encoding is an extension of the one proposed by [Rath et al., 2022].

$\sigma(l_i) = m_j$ means that the substitution $\sigma_{i,j}$ used to bind l_i to m_j is compatible with the other substitutions.

SAT variables - Example

$$L = p(x_1, x_2) \lor p(f(x_2), x_3)$$
$$M = \neg p(f(y), d) \lor p(g(y), c) \lor \neg p(f(c), e)$$

- $b^{-}_{1,1} \Leftrightarrow \{x_1 \mapsto f(y), x_2 \mapsto d\} \subseteq \sigma$

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Constraint to SAT Encoding - Completeness

The constraints can be encoded using a simple procedure. For example the completeness constraint:

$$\forall i.\, \exists j.\, \big(\sigma(l_i) = \neg m_j \vee \sigma(l_i) = m_j\big)$$

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$$\forall i.\, \exists j.\, \big(b^-_{i,j} \vee b^+_{i,j}\big)$$

$$\bigwedge_i \bigvee_j b^-_{i,j} \vee b^+_{i,j}$$

Constraint to SAT Encoding - Completeness

The constraints can be encoded using a simple procedure. For example the completeness constraint:

$$\forall i.\, \exists j.\, \big(\sigma(l_i) = \neg m_j \lor \sigma(l_i) = m_j\big)$$

$$\forall i.\, \exists j.\, \big(b^-_{i,j} \lor b^+_{i,j}\big)$$

$$\bigwedge_i \bigvee_j b^-_{i,j} \lor b^+_{i,j}$$

$$\bigwedge_i \bigvee_j b_{i,j}$$

SR Direct Encoding

SAT-based compatibility $$\bigwedge_i \bigwedge_j [b_{i,j} \Rightarrow \sigma_{i,j} \subseteq \sigma]$$

SAT-based existence $$\bigvee_i \bigvee_j b^-_{i,j}$$

SAT-based uniqueness $$\bigwedge_j \bigwedge_i \bigwedge_{i' \geq i} \bigwedge_{j' > j} \neg b^-_{i,j} \vee \neg b^-_{i',j'}$$

SAT-based completeness $$\bigwedge_i \bigvee_j b_{i,j}$$

SAT-based coherence $$\bigwedge_j \bigwedge_i \bigwedge_{i'} \neg b^+_{i,j} \vee \neg b^-_{i',j}$$

Structuring Variables

We define the following SAT variables:

- c_j is true iff m_j is the resolution literal.

$$c_j \Leftrightarrow \exists i.\, \sigma(l_i) = \neg m_j$$

.

Illustration

$$\begin{aligned}
c_1 &\Leftrightarrow b^-_{1,1} \vee \ldots \vee b^-_{n,1} \\
&\Leftrightarrow \sigma(l_1) = \neg m_1 \vee \ldots \vee \sigma(l_n) = \neg m_1 \\
c_2 &\Leftrightarrow b^-_{1,2} \vee \ldots \vee b^-_{n,2} \\
&\Leftrightarrow \sigma(l_1) = \neg m_2 \vee \ldots \vee \sigma(l_n) = \neg m_2 \\
&\vdots
\end{aligned}$$

SR Indirect Encoding

SAT-based compatibility
$$\bigwedge_i \bigwedge_j [b_{i,j} \Rightarrow \sigma_{i,j} \subseteq \sigma]$$

Structurality
$$\bigwedge_j \left[\neg c_j \vee \bigvee_i b^-_{i,j}\right] \wedge \bigwedge_j \bigwedge_i \left(c_j \vee \neg b^-_{i,j}\right)$$

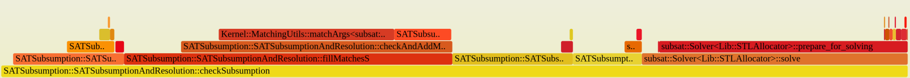
Revised existence
$$\bigvee_j c_j$$

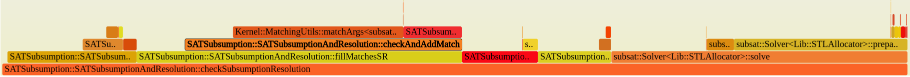
Revised uniqueness
$$AMO(\{c_j, j = 1, ..., |M|\})$$

Revised completeness
$$\bigwedge_i \bigvee_j b_{i,j}$$

Revised coherence
$$\bigwedge_j \bigwedge_i \left(\neg c_j \vee \neg b^+_{i,j}\right)$$

Setting up is Expensive





The setup time takes a significant portion of the total runtime. We can reduce the setup time by setting up both subsumption and SR at the same time.

Optimized Forward Loop

```
procedure Simplify(F, M)
    for L ∈ F \ {M} do
        if checkS(L, M) then
            F ← F \ {L}
            return ⊤
    for L ∈ F \ {M} do
        M* ← checkSR(L, M)
        if M* ≠ ⊥ then
            F ← F \ {L} ∪ {M*}
            return ⊤
    return ⊥
```

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        if M* ≠ ⊥ then
            F ← F \ {L} ∪ {M*}
            return ⊤
    return ⊥
```

```
procedure Simplify*(F, M)
    M* ← ⊥
    for L ∈ F \ {M} do
        if checkS(L, M) then
            F ← F \ {L}
            return ⊤
        if M* = ⊥ then
            M* ← checkSR(L, M)
    if M* ≠ ⊥ then
        F ← F \ {L} ∪ {M*}
        return ⊤
    return ⊥
```

Results - Graph

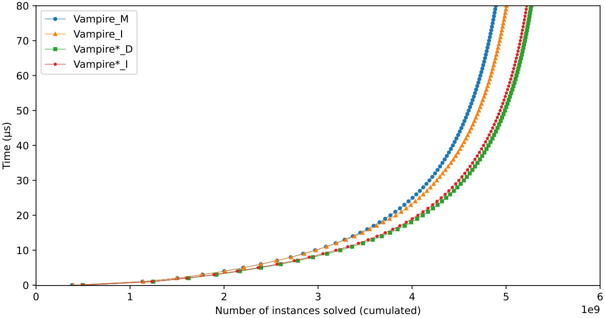

Figure: Comparison of the cumulative number of forward simplification loops solved by the different configurations of Vampire. The graph shows all the loops performed on all the TPTP problems.

Results - Tables

Prover	**Average**	**Std. Dev.**	**Speedup**
Vampire_M	$42.63\,\mu s$	$1609.06\,\mu s$	$0\,\%$
Vampire_I	$40.13\,\mu s$	$1554.52\,\mu s$	$6.2\,\%$
Vampire_D^*	$34.39\,\mu s$	$1047.85\,\mu s$	$23.9\,\%$
Vampire_I^*	$34.55\,\mu s$	$250.25\,\mu s$	$23.4\,\%$

Table: Average and standard deviation of the runtime of forward simplification loop on the TPTP problems.

Prover	**Total Solved**	**Gain/Loss**
Vampire_M	10 555	baseline
Vampire_D^*	10 667	$(+141, -29)$
Vampire_I^*	10 658	$(+133, -30)$

Table: Number of TPTP problems solved by the different configurations of Vampire. The options `-sa otter -av off -t 60` were used for all runs.

Future Work

- Heuristically choose between direct and indirect encoding
- Extend technique to subsumption demodulation
- Investigate the drop in variance.
- Extend subsumption resolution to use an m.g.u. for the resolution literal.

Conclusion

- We have introduced a new method for subsumption resolution.
- SAT-based methods harness the power of modern SAT solvers.
- The setup time of the SAT-based methods is significant. However, we can reduce it by combining the setup of subsumption and SR.
- SAT-based methods are competitive with the state of the art.
- SAT-based methods are also very flexible and can be fine-tuned easily.

References

Rath, J., Biere, A., and Kovács, L. (2022).
First-Order Subsumption via SAT Solving.
In *FMCAD*, page 160.